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Banach Spaces Antiproximinal in Their Biduals

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1. INTRODUCTION

In this note, it is shown that most classical Banach spaces can be renormed (with an equivalent norm) so that they are antiproximinal in their bidual spaces. It is further shown that for many classical spaces X, there is a norm so that $\mathscr{K}(X)$ is not proximinal in $\mathscr{B}(X)$.

A subspace M is *proximinal* in a Banach space X if every x in X has a closest approximant in M. It is called *antiproximinal* if the only vectors with closest approximants are the elements of M. The consideration of whether X is proximinal in X^{**} was first studied in [1], where it was shown to hold for most classical spaces. A case of particular interest is that of the compact operators $\mathscr{K}(X)$ as a subspace of $\mathscr{B}(X)$, which is frequently identified with $\mathscr{K}(X)^{**}$ [3]. For $\mathscr{H}$ a Hilbert space, $\mathscr{K}(\mathscr{H})$ is well known to be proximinal in $\mathscr{B}(\mathscr{H})$ [5]. However, Holmes and Kripke [7] showed that $\mathscr{H}$ can be renormed so that $\mathscr{K}(\mathscr{H}, |\cdot|)$ is not proximinal in $\mathscr{B}(\mathscr{H}, |\cdot|)$. They also showed how to renorm c_0 so as to be antiproximinal in its second dual. Blatter and Seever [2] showed that the disc algebra A is not proximinal in A^{**}. However, it remains an open question as to whether it is proximinal in H^{∞}. It is also known that $\mathscr{K}(l^p)$ is proximinal in $\mathscr{B}(l^p)$ for $1 \leqslant p < \infty$ [6, 9]. I have heard that Y. Benyamini and R. K. Lin have shown that $\mathscr{K}(L^p)$ is not proximinal in $\mathscr{B}(L^p)$ for $1 < p < \infty$, $p \neq 2$ [11].

When X has a Schauder basis (and somewhat more generally), X and $\mathscr{K}(X)$ can be renormed to be antiproximinal in their second duals. If X is $L^p(\mu)$, $1 < p < \infty$, for some measure μ (other than the sum of finitely many atoms) or $C(K)$, where K is an infinite, compact metric space, then X can be renormed so that $\mathscr{K}(X)$ is not proximinal in $\mathscr{B}(X)$. Of course, the identity map I always has 0 as its closest approximant, so $\mathscr{K}(X)$ cannot be antiproximinal in $\mathscr{B}(X)$. Finally, we give an example of a separable, reflexive Banach space X such that $\mathscr{K}(l^1, X)$ is not proximinal in $\mathscr{B}(l^1, X)$.

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2. Antiproximinality

Definition 2.1. A Banach space X will be said to have the *Projection Approximation Property* (*P.A.P.*) if there is an increasing sequence P_n of commuting, finite rank idempotents in $\mathscr{B}(X)$ tending strongly to the identity (i.e., $\lim_{n\to\infty} P_n x = x$ for all x in X.)

The Banach–Steinhaus theorem shows that the sequence P_n is always bounded. So it readily follows that when X has P.A.P., it also has the bounded approximation property (B.A.P.). Furthermore, it will be shown below that X with P.A.P. can be renormed to have the metric approximation property (M.A.P.). Many Banach spaces have P.A.P. In particular, if X has a Schauder basis, the basic projections provide the desired sequence. If X is reflexive and has P.A.P., then so does X^*. For then P_n^* is a sequence of idempotents, and the Hahn–Banach theorem can be used to show that P_n^* tends strongly to I_{X^*}.

It will be of particular interest for us to know when spaces of compact operators have P.A.P.

Lemma 2.2. *If X and Y have P.A.P. and X is reflexive, then $\mathscr{K}(X, Y)$ has P.A.P.*

Proof. Let P_n and Q_n be sequences for X and Y satisfying Definition 2.1. For K in $\mathscr{K}(X, Y)$, define

$$R_n(K) = Q_n K P_n.$$

It is clear that R_n is an increasing sequence of commuting finite rank idempotents. Since K is compact, the unit ball has compact image. So it follows that

$$\lim_{n\to\infty} Q_n K = K$$

By the remarks preceding the proof, P_n^* provides a P.A.P. sequence for X^*. So by the same reasoning,

$$\lim_{n\to\infty} P_n^* K^* = K^*$$

so that KP_n tends to K. Thus

$$K - R_n(K) = (K - Q_m K) + Q_n(K - KP_n)$$

tends to zero in norm as n tends to infinity. ∎

It is also the case that $\mathscr{K}(c_0)$ has P.A.P., but $\mathscr{K}(l^1)$ does not because it is not separable.

The main theorem of this paper can now be stated.

THEOREM 2.3. *If X has P.A.P., then X has an equivalent norm* $|\cdot|$ *such that* $(X, |\cdot|)$ *is antiproximinal in* $(X, |\cdot|)^{**}$.

This immediately yields renormings of spaces with Schauder bases, and their spaces of compact operators so as to be antiproximinal in their second duals. For example, $C(K)$ when K is an infinite compact metric space, $L^1(\mu)$ when μ is a σ-finite measure space,

$$\mathscr{K}(l^p) \text{ and } \mathscr{K}(L^p(0,1)), \qquad 1<p<\infty, \qquad \text{and} \qquad \mathscr{K}(c_0).$$

LEMMA 2.4. *If* $(X, \|\cdot\|$ *has P.A.P., then X has an equivalent norm* $|||\cdot|||$ *such that* $|||P_n|||=1$ *and* $|||I-P_n|||\leqslant 1$. *In particular,* $(X, |||\cdot|||)$ *has the metric approximation property.*

Proof. Let

$$|||x||| = \sup_{0\leqslant n<m<\infty} \|(P_n-P_m)x\|$$

where $P_0=0$ by convention. As $\sup\|P_n\|<\infty$ and $P_n x$ tends to x for all x in X, there is a finite C so that

$$\|x\| \leqslant |||x||| \leqslant C\|x\|.$$

Also, since $P_m P_k = P_{\min(m,k)}$,

$$\begin{aligned} |||P_k x||| &= \sup_{0\leqslant n<m<\infty} \|(P_m-P_n)P_k x\| \\ &= \sup_{0\leqslant n<m\leqslant k} \|(P_m-P_n)x\| \leqslant |||x||| \end{aligned}$$

and

$$\begin{aligned} |||(I-P_k)x||| &= \sup_{0\leqslant n<m<\infty} \|(P_m-P_n)(I-P_k)x\| \\ &= \sup_{k\leqslant n<m<\infty} \|(P_m-P_n)x\| \leqslant |||x|||. \end{aligned}$$

So $|||P_k|||\leqslant 1$ *and* $|||I-P_k|||\leqslant 1$. ∎

From now on, we shall always assume that $(X, \|\cdot\|)$ satisfies Definition 2.1 with projections P_n of norm one with $\|I-P_n\|\leqslant 1$ as well. Let $(Y, \|\cdot\|)$ be an arbitrary Banach space, and suppose that

$$T\colon (X, \|\cdot\|) \to (Y, \|\cdot\|)$$

is a compact operator. Define a new norm on X and X^{**} by

$$|x| = \|x\| + \|Tx\|, \qquad x \in X$$
$$|u| = \|u\| + \|T^{**}u\|, \qquad u \in X^{**}.$$

Since $\|u\| \leqslant |u| \leqslant (1 + \|T\|)\|u\|$, one has $(X, |\cdot|)$ equivalent to $(X, \|\cdot\|)$ and $(X^{**}, |\cdot|)$ equivalent to $(X^{**}, \|\cdot\|)$.

LEMMA 2.5. *In the situation described above,* $(X, |\cdot|)^{**} = (X^{**}, |\cdot|)$.

Proof. Let $|||\cdot|||$ be the norm on $(X, |\cdot|)^{**}$. Fix u in X^{**}. Since the unit ball of $(X, |\cdot|)$ is weak* dense in the unit ball of $(X^{**}, |||\cdot|||)$, there is a net x_α in X such that

$$|x_\alpha| = |||\hat{x}_\alpha||| \leqslant |||u||| \qquad \text{and} \qquad \hat{x}_\alpha \xrightarrow{w^*} u$$

where $\hat{x}$ is the canonical image of x in X^{**}. Since T^{**} is weak* continuous, $(Tx_\alpha) = T^{**}\hat{x}_\alpha$ tends to $T^{**}u$ in the weak* topology (indeed, in norm). So

$$|u| = \|u\| + \|T^{**}u\| \leqslant \underline{\lim}\|x_\alpha\| + \|T^{**}\hat{x}_\alpha\|$$
$$= \lim |x_\alpha| = |||u|||.$$

Conversely, one can choose the net x_α in X so that

$$\|x_\alpha\| \leqslant \|u\| \qquad \text{and} \qquad \hat{x}_\alpha \xrightarrow{w^*} u$$

By the fact that T^{**} is the dual of a compact operator, $Tx_\alpha = T^{**}x_\alpha$ tends to $T^{**}u$ in norm. Whence,

$$|||u||| \leqslant \underline{\lim}|||\hat{x}_\alpha||| = \lim \|x_\alpha\| + \|Tx_\alpha\|$$
$$= \|u\| + \|T^{**}u\| = |u|. \quad \blacksquare$$

Remark. It occurs to me that the compactness of T is probably irrelevant. However, the proof given above relies heavily on this property.

LEMMA 2.6. *In the situation described above, if* $\lim_{n\to\infty} \|T(1-P_n)\| = 0$,

$$\lim_{n\to\infty} |P_n| = 1 = \lim_{n\to\infty} |I - P_n|.$$

Proof. For x in X,

$$\frac{|P_n x|}{|x|} = \frac{\|P_n x\| + \|TP_n x\|}{\|x\| + \|Tx\|} \leqslant \frac{\|x\| + \|Tx\| + \|T(1-P_n)\|\,\|x\|}{\|x\| + \|Tx\|}$$
$$\leqslant 1 + \|T(1-P_n)\|$$

and

$$\frac{|(I-P_n)x|}{|x|} \leqslant \frac{\|(I-P_n)x\| + \|T(I-P_n)x\|}{\|x\|} \leqslant 1 + \|T(1-P_n)\|.$$

Both of these terms tends to 1 uniformly in x as n tends to infinity. ∎

Now for u in X^{**}, define

$$d(u) = \inf_{x \in X} \|u-x\| \qquad \text{and} \qquad d'(u) = \inf_{x \in X} |u-x|.$$

LEMMA 2.7. *Assume the hypotheses of Lemma 2.6. For all u in X^{**},*

$$d(u) = \lim_{n \to \infty} \|(I-P_n^{**})u\| = \lim_{n \to \infty} |(I-P_n^{**})u| = d'(u).$$

Proof. Since P_n has finite rank, P_n^{**} has the same range as P_n and thus $P_n^{**}u$ belongs to X. So

$$d(u) \leqslant \|(I-P_n^{**})u\|$$

for all n. Conversely, let x belong to X and note that

$$\lim_{n \to \infty} \|(I-P_n^{**})u\| \leqslant \lim_{n \to \infty} \|(I-P_n^{**})(u-x)\| + \|(I-P_n)x\|$$

$$\leqslant \|u-x\| + \lim_{n \to \infty} \|(I-P_n)x\| = \|u-x\|.$$

Taking the infimum over X yields

$$\lim_{n \to \infty} \|(I-P_n^{**})u\| = d(u).$$

Using Lemma 2.6, one similarly obtains

$$\lim_{n \to \infty} |(I-P_n^{**})u| = d'(u).$$

Finally, since $\|T^{**}(I-P_n^{**})\| = \|T(1-P_n)\|$ tends to zero,

$$d'(u) = \lim_{n \to \infty} |(I-P_n^{**})u| = \lim_{n \to \infty} \|(I-P_n^{**})u\| + \|T^{**}(I-P_n^{**})u\|$$

$$= \lim_{n \to \infty} \|(I-P_n^{**})u\| = d(u). \quad ∎$$

Proof of Theorem 2.3. First, use Lemma 2.4 to renorm X so that the conclusions of that Lemma apply. Let $Y = X \hat{\otimes} l^1 = \{(x_n)\colon\ x_n \in X, \sum \|x_n\| < \infty\}$. Given $\varepsilon > 0$, define a compact operator T from X into Y by

$$Tx = (2^{-n}\varepsilon P_n x).$$

It is readily apparent that T is the limit of finite rank operators and is injective. Furthermore,

$$\|T(I-P_k)x\| = \sum_{n=1}^{\infty} 2^{-n}\varepsilon \|P_n(I-P_k)x\|$$
$$\leqslant \sum_{n=k+1}^{\infty} 2^{-n}\varepsilon \|x\| = 2^{-k}\varepsilon \|x\|.$$

Now $|x| = \|x\| + \|Tx\|$ is defined as above. By Lemma 2.5, $(X, |\cdot|)^{**} = (X^{**}, |\cdot|)$. Also, since T is injective, so is T^{**}.

Now let u belong to X^{**}. Suppose x belongs to X and $d'(u) = |u-x|$. Then

$$\|u-x\| \leqslant |u-x| = \|u-x\| + \|T^{**}(u-x)\| = d'(u) = d(u) \leqslant \|u-x\|.$$

Hence $T^{**}(u-x)=0$, and thus $u=x$ belongs to X. So $(X, |\cdot|)$ is antiproximinal in $(X^{**}, |\cdot|)$. ∎

3. Compact Operators

In this section, the renorming argument of [7] is modified to apply to all l^p spaces, $1 < p < \infty$. Then well known imbedding techniques give renormings in many situations so that $\mathscr{K}(X)$ is not proximinal in $\mathscr{B}(X)$.

Lemma 3.1. *Suppose $(Y, \|\cdot\|)$ has P.A.P. and the projections P_n satisfy* $\lim_{n\to\infty} \|I-P_n\| = 1$. *Then for all T in $\mathscr{B}(X, Y)$,*

$$\|T\|_e = \inf_{K \in \mathscr{K}(X,Y)} \|T-K\| = \lim_{n\to\infty} \|(I-P_n)T\|$$

Proof. Since $P_n T$ is compact, for any K in $\mathscr{K}(X, Y)$,

$$\|T\|_e \leqslant \lim_{n\to\infty} \|T-P_nT\| \leqslant \lim_{n\to\infty} \|(I-P_n)(T-K)\| + \|(I-P_n)K\|$$
$$\leqslant \|T-K\| + \lim_{n\to\infty} \|(I-P_n)K\| = \|T-K\|.$$

Now take the infimum over all compact operators. ∎

Let Y be as in Lemma 3.1, and let $|\cdot|$ be the norm

$$|y| = \|y\| + \|Ty\|$$

constructed, as in the proof of Theorem 2.3, so that T is injective and

$$\lim_{n\to\infty} \|T(1-P_n)\| = 0$$

For A in $\mathscr{B}(X, Y)$, let $\|A\|$ and $|A|$ be the norms of A as an operator from X into $(Y, \|\cdot\|)$ and $(Y, |\cdot|)$, respectively. Similarly, define $\|T\|_e$ and $|T|_e$. Then one has:

COROLLARY 3.2. *In the situation just described,*

$$\|A\|_e = |A|_e \quad \text{for all } A \text{ in } \mathscr{B}(X, Y).$$

Proof. Clearly, $\|B\| \leqslant |B|$ for all B in $\mathscr{B}(X, Y)$. So by Lemmas 2.6 and 3.1,

$$\begin{aligned} \|A\|_e \leqslant |A|_e &= \lim_{n \to \infty} |(I - P_n)A| \\ &\leqslant \lim_{n \to \infty} \|(I - P_n)A\| + \|T(I - P_n)\| \, \|A\| \\ &= \lim_{n \to \infty} \|(I - P_n)A\| = \|A\|_e. \quad \blacksquare \end{aligned}$$

THEOREM 3.3. *Let X be a Banach space with P.A.P.. Then X has an equivalent norm $|\cdot|$ so that for all $1 < p < \infty$, $\mathscr{K}(l^p, (X, |\cdot|))$ is antiproximinal in $\mathscr{B}(l^p, (X, |\cdot|))$.*

Proof. Let $\|\cdot\|$ be a norm on X as provided by Lemma 2.4, so that $\|P_n\| = 1 = \|I - P_n\|$ for all the projections $\{P_n\}$. Let $|\cdot|$ be the norm

$$|x| = \|x\| + \|Tx\|$$

as constructed in the proof of Theorem 2.3. Also, let Q_n be the standard basis projections on $\text{span}\{e_1, ..., e_n\}$ in l^p for some p in $(1, \infty)$.

Now let A be any bounded operator from l^p into $(X, |\cdot|)$ with $|A|_e = 1$. It suffices to prove that $|A| > 1$. Since $A \neq 0$, one can choose an integer n_0 so that

$$x_0 = Ae_{n_0} \neq 0.$$

Thus $\delta = \|Tx_0\| > 0$ also. Let $\varepsilon = \delta^{q/p}$, where $1/p + 1/q = 1$. Since $(1 - P_n) A(1 - Q_n)$ is a finite rank perturbation of A, by Corollary 3.2 $\|(I - P_n) A(I - Q_n)\| \geqslant 1$. So there are unit vectors $y_n = (1 - Q_n) y_n$ such that

$$\lim_{n \to \infty} \|(I - P_n) Ay_n\| = 1.$$

Now for $n > n_0$,

$$\|y_n + \varepsilon e_{n_0}\| = (1 + \varepsilon^p)^{1/p} = (1 + \delta^q)^{1/p}$$

and

$$|A(y_n + \varepsilon e_{n_0})| = |Ay_n + \varepsilon x_0|$$
$$= \|Ay_n + \varepsilon x_0\| + \|TAy_n + \varepsilon Tx_0\|.$$

Since y_n tends to zero in the weak topology on l^p,

$$\lim_{n \to \infty} \|TAy_n\| = 0.$$

Thus

$$\overline{\lim_{n \to \infty}} |A(y_n + \varepsilon e_{n_0})|$$
$$\geqslant \lim_{n \to \infty} \|(I - P_n)Ay_n\| - \varepsilon \|(I - P_n)x_0\| + \varepsilon \|Tx_0\| - \|TAy_n\|$$
$$= 1 + \varepsilon\delta = 1 + \delta^{q/p + q/q} = 1 + \delta^q.$$

It follows that

$$|A| \geqslant \frac{1 + \delta^q}{(1 + \delta^q)^{1/p}} = (1 + \delta^q)^{1/q}.$$

So no non-zero operators have norm equal to their essential norm. ∎

Remark 3.4. This theorem holds with l^p replaced by c_0. To see this, note that $\|y_n + \varepsilon e_{n_0}\| = 1$, and y_n still tends weakly to zero. So the same estimates are valid.

COROLLARY 3.5. *For each $1 < p < \infty$, there is a norm $|\cdot|$ on l^p so that $\mathcal{K}(l^p)$ is not proximinal in $\mathcal{B}(l^p)$. Similarly, this holds for c_0.*

Proof. The space l^p is isometrically isomorphic to $l^p \oplus_p l^p$ with $\|(x, y)\| = (\|x\|_p^p + \|y\|_p^p)^{1/p}$. Put an equivalent norm on $l^p \oplus l^p$ by

$$|(x, y)| = (\|x\|_p^p + |y|^p)^{1/p}$$

where $|\cdot|$ is the norm constructed in Theorem 3.2. The projection P onto the first summand satisfies $\|P\| = \|I - P\| = 1$. If T is any non-zero operator of the form $T = (I - P)\,TP$ and K is compact, then

$$\|T - K\| \geqslant \|(I - P)(T - K)P\| = \|T - (1 - P)\,KP\|.$$

But $(I - P)\,KP$ and T can be thought of as operators from l^p to $(l^p, |\cdot|)$, so the norm is strictly greater than $\|T\|_e$. Hence T has no closest approximant.

The case of c_0 follows from Remark 3.4. ∎

COROLLARY 3.5. *If an infinite dimensional space X is $L^p(\mu)$ for a Borel measure μ and $1 < p < \infty$ or $X = C(K)$ for any infinite compact metric space K, then X has an equivalent norm $|\cdot|$ so that $\mathscr{K}(X, |\cdot|)$ is not proximinal in $\mathscr{B}(X, |\cdot|)$.*

Proof. Provided $X = L^p(\mu)$ is infinite dimensional, it contains an isometric copy of l^p which is the range of a norm one projection P [8]. Likewise, K contains a sequence x_n with limit x_0, and the restriction R to $A = \{x_n, n \geqslant 0\}$ is a contractive map of $C(K)$ onto $c \simeq c_0$. By Michael's Selection Theorem [10], there is an isometric linear imbedding J of c into $C(K)$ so that $P = RJ$ is a contractive projection onto a copy of c.

In each case, let $|\cdot|$ be a norm on l^p (or c_0) so that $\mathscr{K}(l^p)$ (or $\mathscr{K}(c_0)$ is not proximinal in $\mathscr{B}(l^p)$ (*or* $\mathscr{B}(c_0)$). Put a norm on X by

$$|||x||| = \|x\| + (|Px| - \|Px\|)$$

Since $|Px| - \|Px\|$ has the form $\|TPx\|$, it is a seminorm, so $|||\cdot|||$ is a norm such that $|||Px||| = |Px|$. Now if T is an operator on l^p or c_0 for which there is no closest compact approximant, then $\tilde{T} = PTP$ gives an operator on X with the same property. Following the previous proof, if K is compact

$$\|\tilde{T} - K\| \geqslant \|T - PKP\| > \|T\|_e. \quad \blacksquare$$

In view of Theorem 3.3, one might ask about the proximinality of $\mathscr{K}(l^1, X)$ in $\mathscr{B}(l^1, X)$. However, the situation here is quite different. An operator T in $\mathscr{B}(l^1, X)$ is determined by the sequence $x_n = Te_n$ in X. Any bounded sequence in X yields a bounded operator, and

$$\|T\| = \sup \|x_n\|.$$

It is shown in [4] that a best approximation of T by compact operators is equivalent to finding a best approximation of the image of the unit ball under T by a compact set in X. In [4, 9], it is shown that if X is uniformly rotund, then $\mathscr{K}(l^1, X)$ is proximinal in $\mathscr{B}(l^1, X)$.

In contrast to Theorem 3.2, $\mathscr{K}(l^1, X)$ is never antiproximinal in $\mathscr{B}(l^1, X)$. To see this, let x_n be any sequence dense in the unit ball of X (or the unit ball of an infinite dimensional, separable subspace of X if X is not separable). Then the operator T defined by $Te_n = x_n$ has $\|T\| = \|T\|_e = 1$. It is possible, though, to find even reflexive Banach spaces X such that $\mathscr{K}(l^1, X)$ is not proximinal in $\mathscr{B}(l^1, X)$. In view of the remarks in the preceding paragraph, this also yields an example of a closed bounded convex set in X without best compact approximant.

EXAMPLE 3.6. Let $X = \bigoplus_{l^2} \sum_{n=2}^{\infty} l^n$ denote the l^2 direct sum of the l^p spaces for $p = 2, 3, 4, \ldots$. This is a reflexive Banach space with dual

$X^* = \oplus_{l^2} \sum_{n=2}^{\infty} l^{n/(n-1)}$. Let $\{e_{n,m}, n \geqslant 2, m \geqslant 1\}$ be a standard basis for l^1, and let $\{f_{n,m}, m \geqslant 0\}$ be a standard basis for l^n, $n \geqslant 2$. Define $T: l^1 \to X$ by

$$Te_{n,m} = f_{n,0} + f_{n,m}.$$

Let P_n be the obvious contractive projection from X onto l^n.

It is clear that $\|P_n T\| = 2^{1/n}$ and $\|T\| = \sqrt{2}$. For each $n \geqslant 2$, let K_n be the rank operator given by

$$K_n e_{k,m} = \delta_{nk} f_{n,0}$$

where δ_{nk} is the Kronecker delta function. It is readily apparent that $\|P_n T - K_n\| = 1$. Furthermore, Lemma 3.1 shows that

$$\|P_n T\|_e = 1.$$

Thus

$$\left\| T - \sum_{n-1}^{N-1} K_n \right\| = 2^{1/N}$$

and hence $\|T\|_e = 1$.

It will be shown that T has no best compact approximant. Suppose K is any compact operator from l^1 into X. Then it follows that

$$\lim_{n \to \infty} \|P_n K\| = 0.$$

Thus there is an integer N so that $\|P_N K\| \leqslant \frac{1}{2}$. Since

$$\|T - K\| \geqslant \|P_N T - P_N K\|,$$

it suffices to show that if C is a compact operator from l^1 into l^n such that $\|P_n T - C\| = 1$, then $\|C\| \geqslant 1$. Let us write T_n for $P_n T$ as an operator into l^n.

Fix $0 < r < 1$, and let C be an operator from l^1 to l^n with $\|C\| \leqslant r$ and $\|T_n - C\| \leqslant 1$. For simplicity of notation, write e_m for $e_{n,m}$ and f_m for $f_{n,m}$. Let f_m^* be the biorthogonal basis for $l^{n/(n-1)} = (l^n)^*$. Let $x_m = Te_m = f_0 + f_m$ and $y_m = Ce_m$. Since

$$|f_0^*(y_m)| \leqslant \|C\| \leqslant r,$$

one has

$$|f_0^*(x_m - y_m)| \geqslant 1 - r$$

whence

$$|f_m^*(x_m - y_m)| \leqslant (1-(1-r)^n)^{1/n}.$$

From this, it follows that

$$|f_m^*(y_m)| \geqslant 1-(1-(1-r)^n)^{1/n} = \delta > 0.$$

Hence by Lemma 3.1, if Q_m is the projection in l^n onto span $\{f_0, ..., f_m\}$,

$$\|C\|_e = \lim_{n \to \infty} \|(I-Q_m)C\| \geqslant \delta > 0.$$

So C is not compact, and the argument is complete.

Although X is not uniformly rotund, it is easy to verify that it is locally uniformly rotund in the sense that for all x in the unit ball of X, and $0 < \varepsilon < 1$

$$\delta(x, \varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|y\| = 1, \|x-y\| \geqslant \varepsilon \right\}$$

is strictly positive. This clearly shows the limits of the results of [4, 9].

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